CARe Practitioners' Track Pricing and Use of Aggregate Distributions

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Computing Aggregates

- General philosophy
- Counts x Severity = Loss Pick
- Freq. Dist ^ Sev. Dist = Aggregate Dist
- ^ = Compound Distribution
- Need to select frequency and severity distributions

Computing Aggregates

- **Fourier Transform Based Methods**
  - Continuous Fourier Transform
    - Heckman-Meyers
  - Discrete Fourier Transform
    - Fast Fourier Transform
- **Method of Moments**
- **Panjer-Wilmot Recursive Method**
- **Simulation**

Computing Aggregates

- **Method of Moments**
  - Mean and variance
    - Lognormal, gamma, other two parameter distributions
  - Mean, variance, skewness
    - 3 parameter or shifted lognormal
    - 3 parameter or shifted gamma
      - Bowers, Gerber, Hickman, Jones, Nesbitt, ...
    - Generalized gamma
  - Mean, variance, skewness, kurtosis
    - Generalized beta
      - Very flexible, but hard to fit

Computing Aggregates

- **Method of Moments**
  - Moments of severity distributions and frequency distributions are available in literature
    - Moments of **layers** of severity distributions is an exercise in integration
    - Use integration by parts and recursive function calls, rather than deriving a closed form expression
  - For skewness of aggregate see Bowers et al.

Aggregate Distributions and AAD's

- **AAD has non-linear payoff: max(X-k,0)**
  - By Jensen's inequality

    $$E(\max(X-k,0)) > \max(E(X)-k,0)$$

    - Explains why full credit not given for AAD
  - Many other examples of Jensen's < in actuarial science
    - Annuity certain for expected future life vs. a(x)
  - Remembering Jensen
    - Since variance is positive, $E(X^2) > E(X)^2$
- **Aggregate distributions also help actuary figure discount factor**

Example: Loss Pick

- **Counts x Severity = Loss Pick**
- **Counts**
  - Look at trended counts greater than $550K
    - 5% trend, can't look at smaller claims
  - Triangle and development shown on next slide
  - Indicate roughly 75 claims xs $550K per year
  - Trended experience has 261 claims xs $550K, 47 of which are strictly greater than $1M
  - Counts to layer approx. 47 / 261 x 75 = 13.5

Example: Loss Pick

Treaty Period	Development Period 1	2	3	4	5	6
1992	13	36	51	61	78	81
1993	13	40	52	63	74	
1994	7	18	26	31		
1995	7	26	42			
1996	9	24				
1997	9					

Individual Factors

Treaty Period	Development Period 1-2	2-3	3-4	4-5	5-6	6-7
1992	2.77	1.42	1.20	1.28	1.04	
1993	3.08	1.30	1.21	1.17		
1994	2.57	1.44	1.19			
1995	3.71	1.62				
1996	2.67					

Volume Weighted Averages					
	1-2	2-3	3-4	4-5	5-6
All	2.939	1.425	1.202	1.226	1.038
Last 3	2.957	1.429	1.202	1.226	1.038
All (Ex Last)	3.000	1.372	1.204	1.279	
Last 3 (Ex Last)	3.111	1.372	1.204	1.279	
SELECTED	3.000	1.420	1.200	1.250	1.050
LDF	7.716	2.572	1.811	1.509	1.208
Pattern	13.0%	38.9%	55.2%	66.3%	82.8%
Interp. Patt	13.0%	38.9%	55.2%	66.3%	82.8%

Latest	Pattern	Ultimate	l Premium	Frequency per 100M
81	87.0%	93.15	1,000	9.32
74	82.8%	89.36	1,000	8.94
31	66.3%	46.79	1,000	4.68
42	55.2%	76.07	1,000	7.61
24	38.9%	61.73	1,000	6.17
9	13.0%	69.44	1,000	6.94

Average Frequency:	From	To	
	1992	1997	7.28
	1992	1996	7.34

Example: Loss Pick

- **Counts x Severity = Loss Pick**
- **Severity**
  - Select $653,000 from "pivot table" of trended limited severities
  - P/O 2 curve gives severity of $632,000
  - Choose not to develop individual claims
- **ALAE added as flat 20%**

AY	Data	Total	Average
92	Sum of Layer Loss	6,924,909	629,537
	Sum of Count	11	
93	Sum of Layer Loss	13,893,858	731,256
	Sum of Count	19	
94	Sum of Layer Loss	2,708,356	541,671
	Sum of Count	5	
95	Sum of Layer Loss	3,252,011	464,573
	Sum of Count	7	
96	Sum of Layer Loss	2,891,521	722,880
	Sum of Count	4	
97	Sum of Layer Loss	1,000,000	1,000,000
	Sum of Count	1	
Total Sum of Layer Loss		30,670,654	652,567
Total Sum of Count		47	

Example: Loss Pick

- Loss Pick = 13.5 x 653K x 1.2 = $10.6M
- To compute aggregate need to select frequency and severity distributions

Example: Frequency Distribution

- **Many choices for frequency distribution**
- **Poisson good for rare events**
- **Over-dispersion (variance > mean) often makes Poisson a poor choice**
- **Negative Binomial more realistic, models**
  variance = constant x mean
  - Used by Heckman-Meyers (contagion parameter)
  - constant = Variance Multiplier
- **In example, use Negative Binomial with variance = 10 x mean (c=0.67)**

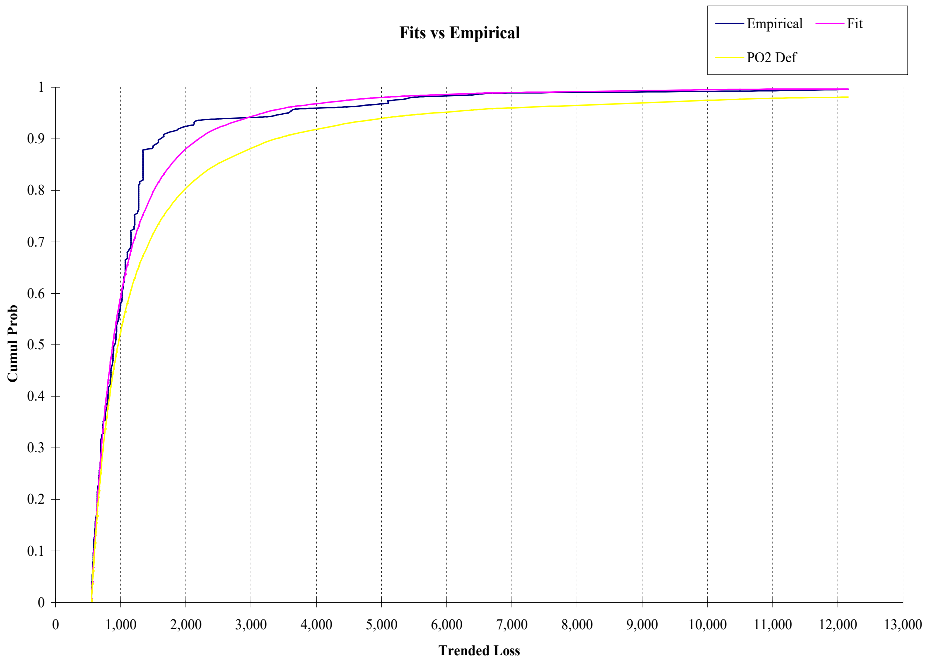
Example: Severity Distribution

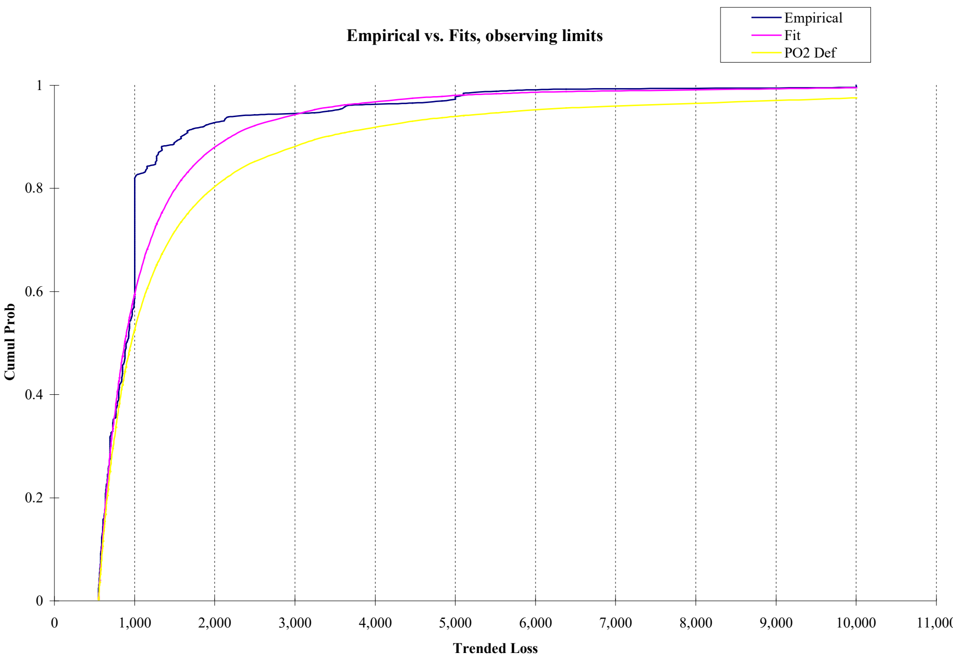
- **Use empirical distribution**
  - Losses trended respecting policy limits
  - Easier than trying to fit severity curves
  - See paper to be presented at DFA seminar on resampling and bootstrapping
  - Most numerical methods use discrete severity distributions and do not require fitted distribution
- **Assume severity given by trended empirical distribution, respecting limits (47 data points)**
- **Can also use smoothed distribution**

Fitting Severity Curves

- **Beware discontinuities at round numbers**
- **Beware trending through limits**
  - Would generate numerous claims just over $1M which lowers estimated severity and distorts aggregate distribution
- **See graphs on next slide**
  - Left hand graph trends through policy limits
  - Right hand graph trends respecting policy limits
    - Only difference is empirical distribution

Fitting Severity Curves





Example: Aggregate from Moments

- **Assume Negative Binomial frequency and trended empirical severity**
- **Moments**
  - Frequency = 13.5 CV = 0.86 skew = 1.64
  - Severity = $784K CV = 0.53 skew = -0.32
  - Aggregate = $10.6M CV = 0.87 skew = 1.64
- **Shifted Lognormal fit**
  - t = -7.76M (decreases skewness)
  - mu = 16.6
  - sigma = 0.48

Example: Aggregate from Moments

- **Loss picks**
  - $5M AAD = $6.5M
  - $7.5M AAD = $4.9M
  - $10M AAD = $3.7M
- **See slide 24 for comparison with FFT method**

DFT Basics

- **Do not have time for thorough review**
- **Recommend the following books:**
  - *The Fast Fourier Transform and its Applications*, by E. Oran Brigham (especially good)
  - *Numerical Recipes in C* by Press, Flannery, Teukolsky, and Vetterling
  - *Fast Transforms: Algorithms, Analyses, Applications* by Elliott and Rao

DFT Basics

- **DFT converts an *n*-point discrete sample of a distribution into an *n*-point sample of the continuous Fourier transform**
- **FFT is a quick method of computing DFT's**
  - See Rao for nice description of method in-terms of factoring matrices
- **Sample regarded as starting at $0**
- ***n* a power of 2 for maximum efficiency, generally between 1,024 and 65,536 in applications**

DFT Basics

- **Computing DFT's**
  - Excel has FFT add-in
    - Tools, Data Analysis, Fourier Analysis
    - Slow, hard to work with complex numbers
  - SAS IML
    - Very fast, but no built in support for complex numbers
    - Can be used in practical application
    - DDE to Excel
  - MATLAB
    - Very fast, built in complex numbers, easy to use
    - DDE / Active X to Excel
  - Other software...

DFT Basics

- **DFT computed as a linear combination of powers of roots of unity**
  - Input gives coefficients
- **First element of DFT is sum of elements of input**
  - If input is discrete severity distribution this equals 1
- **Middle element is real for real input vector**
- **All other terms are complex numbers**
- **Second half of DFT is complex conjugate of first half**

DFT Basics

- **Fourier transform methods based moment generating function identity**

$$M_X(t) = M_N(\log(M_X(t)))$$

where

  - N = frequency random variable
  - X = severity random variable
  - S = aggregate random sum
- **For most frequency distributions $M_N(t)$ is actually a function of e^t**
  - Do not need to compute logs
  - Very important, since that is hard---why?

Simple DFT Example

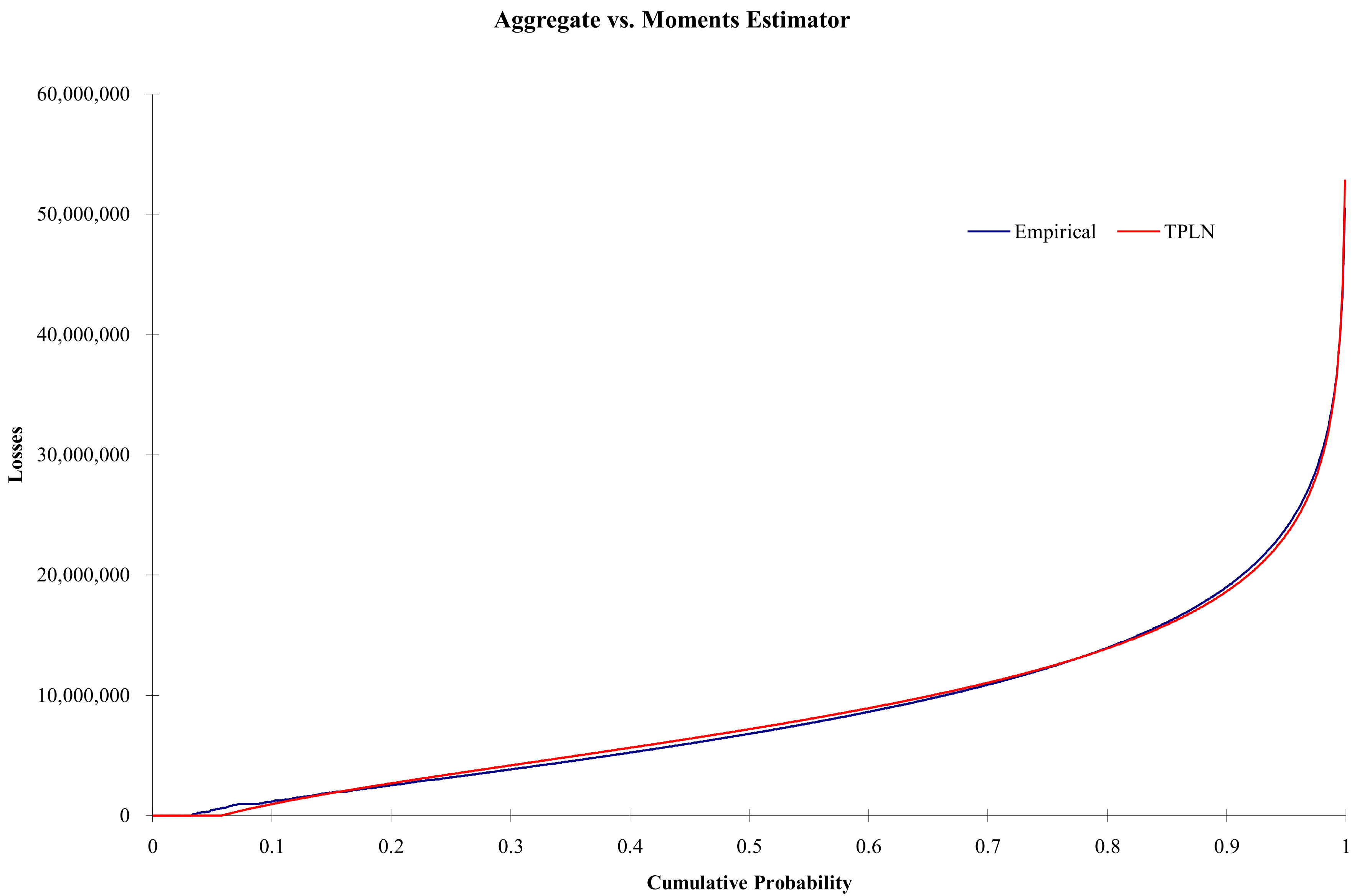
- **If severity distribution is $1 with certainty then aggregate distribution = frequency distribution**
- **Gives method to compute counting distributions**
- **From definition DFT(0,1,0,...,0) = n^{th} roots of unity**
  - Vertices of regular *n*-gon in complex plane
- **Next slide outlines Excel calculation for Poisson distribution with expected value of 5**
  - Excel IMMULT, IMEXP etc.
- **Uses 32 buckets**

Simple DFT Example

N	Sev	FFT(Sample)	exp(5(FFT-1))	IFFT	Actual	Error
0	0	1	1	0.006738	0.006738	-1.2E-13
1	1	0.98078528040323-0.195090322016128i	0.509423855191203-0.752112180031036i	0.033690	0.033690	-4.9E-15
2	0	0.923879532511287-0.38268343236509i	-0.229609511356744-0.643725758779707i	0.084224	0.084224	-8.2E-16
3	0	0.831469612302545-0.555570233019602i	-0.402396127528844-0.153184348985286i	0.140374	0.140374	-3.2E-15
4	0	0.707106781186547-0.707106781186548i	-0.21349217157604+8.87422067842442E-002i	0.175467	0.175467	-1.6E-15
5	0	0.555570233019602-0.831469612302545i	-5.71117635819395E-002+9.21064531448694E-002i	0.175467	0.175467	-1.6E-15
6	0	0.382683432365089-0.923879532511287i	-4.2396539375581E-003+4.54604402340852E-002i	0.146223	0.146223	-2.8E-15
7	0	0.195090322016128-0.98078528040323i	3.4021680408073E-003+1.75446722567033E-002i	0.104445	0.104445	-1.3E-16
8	0	-1i	1.91130077129597E-003+6.4611809388167E-003i	0.065278	0.065278	6.38E-16
9	0	-0.195090322016129-0.98078528040323i	4.83603301812102E-004+2.49389839972149E-003i	0.036266	0.036266	1.34E-15
10	0	-0.382683432365091-0.923879532511286i	-9.23330919027176E-005+9.90057931117126E-004i	0.018133	0.018133	9.57E-15
11	0	-0.555570233019603-0.831469612302545i	-2.20757113108346E-004+3.56023933068327E-004i	0.008242	0.008242	1.81E-14
12	0	-0.707106781186548-0.707106781186547i	-1.81324389075845E-004+7.53710373153621E-005i	0.003434	0.003434	6.25E-14
13	0	-0.831469612302546-0.555570233019601i	-9.8543310422437E-005-3.75135142244478E-005i	0.001321	0.001321	1.42E-13
14	0	-0.923879532511287-0.382683432365089i	-2.23168063640223E-005-6.25666725447613E-005i	4.7174E-04	4.7174E-04	4.69E-13
15	0	-0.980785280403231-0.195090322016127i	2.8027407378568E-005-4.13795982447673E-005i	1.5725E-04	1.5725E-04	1.29E-12
16	0	-1	4.53999297624849E-005	4.9139E-05	4.9139E-05	4.82E-12
17	0	-0.98078528040323+0.195090322016128i	2.80274073785678E-005+4.13795982447674E-005i	1.4453E-05	1.4453E-05	1.44E-11
18	0	-0.923879532511287+0.38268343236509i	-2.23168063640223E-005+6.25666725447613E-005i	4.0146E-06	4.0146E-06	4.86E-11
19	0	-0.831469612302545+0.555570233019602i	-9.8543310422437E-005+3.75135142244478E-005i	1.0565E-06	1.0565E-06	1.22E-10
20	0	-0.707106781186547+0.707106781186548i	-1.81324389075845E-004-7.53710373153621E-005i	2.6412E-07	2.6412E-07	3.58E-10
21	0	-0.555570233019602+0.831469612302545i	-2.20757113108346E-004-3.56023933068327E-004i	6.2886E-08	6.2886E-08	1.9E-10
22	0	-0.382683432365089+0.923879532511287i	-9.23330919027079E-005-9.90057931117127E-004i	1.4292E-08	1.4292E-08	-7.4E-09
23	0	-0.195090322016128+0.98078528040323i	4.83603301812102E-004-2.49389839972149E-003i	3.1070E-09	3.1070E-09	-5.8E-08
24	0	1i	1.91130077129597E-003-6.4611809388167E-003i	6.4729E-10	6.4729E-10	-3.9E-07
25	0	0.195090322016129+0.98078528040323i	3.4021680408073E-003-1.75446722567033E-002i	1.2946E-10	1.2946E-10	-1.9E-06
26	0	0.382683432365091+0.923879532511286i	-4.23965393755858E-003-4.54604402340856E-002i	0.0000E+00	2.4896E-11	
27	0	0.555570233019603+0.831469612302545i	-5.71117635819395E-002-9.21064531448694E-002i	0.0000E+00	4.6104E-12	
28	0	0.707106781186548+0.707106781186547i	-0.213492171576043-8.87422067842451E-002i	0.0000E+00	8.2328E-13	
29	0	0.831469612302546+0.555570233019601i	-0.402396127528846+0.153184348985287i	0.0000E+00	1.4194E-13	
30	0	0.923879532511287+0.382683432365089i	-0.229609511356744+0.643725758779707i	0.0000E+00	2.3657E-14	
31	0	0.980785280403231+0.195090322016127i	0.50942385519121+0.752112180031037i	0.0000E+00	3.8157E-15	

Example: FFT

- **Continue to use trended empirical severity and negative binomial frequency distributions**
- ***n* = 4,096 buckets, each about $25,000 wide**
- **Loss picks**
  - $5M AAD = $6.4M ($6.5M using moments)
  - $7.5M AAD = $4.9M ($4.9M)
  - $10M AAD = $3.7M ($3.7M)
- **Graph on following slide compares cumulative probability functions: DFT vs. shifted lognormal fitted by method of moments**


Aggregate vs. Moments Estimator
Losses
60,000,000
50,000,000
40,000,000
30,000,000
20,000,000
10,000,000
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Cumulative Probability
Empirical
TPLN

Parameter Risk, Sensitivity Testing

- **Inflation**
  - Compare Heckman-Meyer's mixing parameter
  - Measure of unexpected inflation
  - Considers leveraged effect of excess layers and average time to payout
  - Impact of underlying limits becomes an issue
- **Frequency Variance Multiplier**
  - Heckman-Meyers contagion parameter
  - Beta negative binomial
- **Summary on next slide**

Parameter Risk, Sensitivity Testing

	Variance Multiplier (Contaigon Parameter)				
AAD	**VM = 2**	**VM = 3**	**VM = 5**	**VM = 10**	**VM = 15**
5.0M	5.735	5.784	5.975	6.359	6.688
7.5M	3.637	3.815	4.190	4.853	5.358
10.0M	2.028	2.309	2.818	3.674	4.303

	Unexpected Inflation Factor					
AAD	**0.98**	**0.99**	**1.00**	**1.01**	**1.02**	**Avg**
5.0M	4.653	5.452	6.359	7.377	8.510	6.470
7.5M	3.400	4.073	4.853	5.751	6.773	4.970
10.0M	2.473	3.022	3.674	4.440	5.334	3.789

- **Could take probability weighted average over different VM and inflation factors**

Parameter Risk, Sensitivity Testing

- **Use average over unexpected inflation as proxy for mixing parameter**
  - Differentiates high layers from low layers
  - Differentiates long payout from short payout lines
- **Use average over different variance multipliers**
  - Use beta negative binomial to reflect uncertainty in estimation
  - MGF of beta negative binomial is a hypergeometric function not commonly implemented in math programs

Example: Discount Factor

- **Need to assess when payout will reach AAD**
- **Assume payout pattern independent of ultimate loss amount**
  - Hard to do otherwise
  - Area for future research
  - Similar to bond pricing problem
- **Using payout pattern, see when losses hit AAD for various ultimate losses to treaty**
- **Compute PV of reinsured payments**
- **Allow for accelerated payment risk?**

Summary

- **Input assumptions drive differences in results, not computational methods**
- **Moments method works well for moderate and large claim counts**
  - Quick to use
  - Easy to implement
    - Ideal spreadsheet application
  - Accurate answers
    - Use three moments and shifted lognormal
  - Does not work for small claim counts because aggregate distribution is typically not continuous

Summary

- **DFT method**
  - Fast, accurate, flexible
  - Requires some programming to set up efficiently
  - Can be used for complex problems
    - Add distributions from many lines
    - Model cat programs with unique reinstatement provisions
    - Model bivariate distribution of net and gross